数理解析研究所講究録
856巻 1994年 142-150

TORSION FREE THEOREMS FOR HIGHER DIRECT IMAGE SHEAVES OF SEMI-POSITIVE VECTOR BUNDLES

Kensho Takegoshi (竹腰見昭)

College of General Education, Osaka University

§1. In this lecture we discuss some ring theoretic properties of a certain coherent analytic sheaf. Our method is purely analytic and is based on L^2-estimates for the $\bar{\partial}$-operator and a global analysis about semi-positive bundle valued L^2 harmonic forms and bounded plurisubharmonic functions.

Let $f : X \to Y$ be a proper surjective morphism of reduced analytic spaces with X non-singular, $\dim_C X = n$ and $\dim_C Y = m$ respectively. In this talk we discuss the torsion freeness of the the higher direct image sheaves $R^q f_* \Omega^n_X(E)$ for a Nakano semi-positive vector bundle E on X. Particularly we are interested in the sheaves $R^q f_* \Omega^n_X$. It is known that those sheaves are much better behaved than the higher direct image sheaves of the structure sheaf O_X of X. This phenomenon was first verified by Grauert and Riemennschneider [G-R]. They showed that if X is projective algebraic and f is a proper modification, then $R^q f_* \Omega^n_X = 0$ for $q \geqq 1$. Using this vanishing theorem they showed a vanishing theorem for almost positive vector bundles on Moishezon spaces. In this sense one can say that their result gave an intrinsic motive to generalize the vanishing theorem of positive line bundles on compact complex manifolds by Kodaira to various directions in analytic and algebraic geometry. After further contributions and developments by various mathematicians Kollár succeeded to get a large progress with respect to the torsion freeness of $R^q f_* \Omega^n_X$ in [Ko-1] and [Ko-2]. He showed that if X and Y are projective algebraic, then (i) $R^q f_* \Omega^n_X$ is torsion free for $q \geqq 0$ and $R^q f_* \Omega^n_X = 0$ for $q > n - m$, (ii) $H^p(Y, O_Y(A) \otimes R^q f_* \Omega^n_X) = 0$ for any ample line bundle A on Y with $p \geqq 1$ and $q \geqq 0$. On the other hand such a vanishing was first formulated and proved by Ohsawa [Oh] for the case $q = 0$ in a more general situation. Moreover setting $E = f^* A$, he observed that these statements are equivalent to show that the homomorphism $\mu^q_\sigma : H^q(X, \Omega^n_X(jE)) \to H^q(X, \Omega^n_X((j+k)E))$ induced by the tensor product with a holomorphic section σ of the k times tensor product kE of E is injective for any $q \geqq 1$, j and $k \geqq 1$ (cf. [Ko-1], Theorems 1 and 2). In [Ko-2] he discussed the local freeness of $R^q f_* \Omega^n_X$. Later his results are proved by Nakayama [Ny] independently and are generalized by Moriwaki [Mo]. Morihiko Saito also obtained those results as a part of his theory of Hodge modules [Sa].

However the above results are more or less restricted within the projective algebraic category in view of their method. But it seems to be natural to expect that those results are generalized to the category of Kähler manifolds.

The main purpose of this lecture is to report that we can generalize the above results by Kollár to the category of Kähler manifolds. On the other hand it is known that the above torsion freeness property is not necessarily true if X is not Kähler (cf. [Nm]). The principal part of our main result is stated as follows.

Theorem Let $f:(X, \mathcal{O}_X)\to(Y,\mathcal{O}_Y)$ be a proper surjective morphism of irreducible analytic spaces with X non-sigular, $\dim_{\mathbf{C}} X = n$ and $\dim_{\mathbf{C}} Y = m$ respectively.
Suppose X carries a Kähler metric ω_X and $\pi:(E,h)\to(X,\omega_X)$ is a Nakano semi-positive vector bundle of rank r on X. Then

(i) For $q\geqq 1$ there exists a splitting sheaf homomorphism δ^q: $R^q f_*\Omega_X^n(E)\to R^0 f_*\Omega_X^{n-q}(E)$ with $L^q\circ\delta^q = id$ for the sheaf homomorphism $\mathcal{L}^q: R^0 f_*\Omega_X^{n-q}(E)\to R^q f_*\Omega_X^n(E)$ induced by the q-times left exterior product by ω_X.
In particular $R^q f_*\Omega_X^n(E)$ is torsion free for any $q\geqq 0$ and $R^q f_*\Omega_X^n(E)=0$ if $q>n-m$.

(ii) If E is of rank one and the k times tensor product kE of E admits a non-trivial holomorphic section σ on X, then the sheaf homomorphism

$$\mu_\sigma^q : R^q f_*\Omega_X^n(jE)\to R^q f_*\Omega_X^n((j+k)E)$$

induced by the tensor product with σ is injective for any $q\geqq 0$, j and $k\geqq 1$

(iii) If $g: Y\to S$ is a proper surjective morphism of analytic spaces and A is a g-ample line bundle on Y, then $R^p g_*(\mathcal{O}_Y(A)\otimes R^q f_*\Omega_X^n)=0$ for any $p\geqq 1$ and $q\geqq 0$

(iv) Suppose Y is non-singular. Then if f is a regular family outside a normal crossing divisor of Y, $R^q f_* \mathcal{O}_X(K_{X/Y})$ is locally free for any $q \geqq 0$, where $K_{X/Y}$ is the relative canonical line bundle defined by $K_{X/Y} := K_X \otimes f^* K_Y^{-1}$.

§2. In this section we give an outline of the proof of Theorem. Since the problem is local, we fix a point y of Y and replace Y by a connected analytic subset S in a unit ball in C^d with a global coordinate system $(t_1, t_2, \ldots, t_d)$ and $t_l(y) = 0$ for any l. Setting $\psi := \sum_{l=1}^{m} |t_l|^2$ and $\varphi := f^* \psi$ we put $S_c := \{\psi < c\}$ and $X(S_c) := f^{-1}(S_c)$ with $0 < c \ll 1$.
Since f is proper, Grauert's direct image theorem tells us:

(2.1) For any $q \geqq 0$ and $0 < c << 1$ the canonical homomorphism

$$R^q f_* : H^q(X(S_c), \Omega_X^n(E)) \to \Gamma(S_c, R^q f_* \Omega_X^n(E))$$

is a topological isomorphism. In particular $H^q(X(S_c), \Omega_X^n(E))$ is a finitely generated $\mathcal{O}(S_c)$-module and has a structure of a separated topological vector space, where $\mathcal{O}(S_c)$ is the ring of holomorphic functions on S_c.

We fix two small positive numbers a and b with $b < a$ and can assume that the $\bar{\partial}$ closed E-valued smooth (n, q) forms $\{v_1, \ldots, v_k\}$ on $X(S_b)$ generates $H^q(X(S_c), \Omega_X^n(E))$ over $\mathcal{O}(S_c)$ for any $0 < c \leqq a$. By the Kählerity of ω_X, the semi-positivity and (2.1) of (E, h), we can decompose the forms v_k as follows:

Lemma 2.2. If $q \geqq 1$, then

(i) $v_j = u_j + \bar{\partial} w_j$ on $X(S_b)$ with $u_j \in C^{n,q}(X(S_b), E)$ and $w_j \in C^{n,q-1}(X(S_b), E)$ for any j

(ii) $\bar{\partial} u_j = 0, \overline{\partial_h}^* u_j = 0, \bar{\partial} * u_j = 0$ and $< i\, e(\Theta_h) \Lambda u_j, u_j >_h = 0$

on $X(S_b)$ for any j

(iii) $\partial \tau \wedge * u_j = 0$ and $< i\, \partial\bar{\partial} \tau\, \Lambda u_j, u_j > = 0$ on $X(S_b)$

for any j and smooth plurisubharmonic function τ on $X(S_a)$ satisfying $\sup_{x \in X} \{|\tau(x)| + |d\tau(x)|\} < \infty$

where $\overline{\partial_h}^*$ is the formal adjoint operator of $\bar{\partial}$ and $*$ is the Hodge's star operator relative to ω_X.

We denote by $O(S_c)(u_1, \ldots, u_k)$ the set of linear combinations of $\{u_1, \ldots, u_k\}$ with coefficients $O(S_c)$. Then using (ii), (iii) and the integral formula in Proposition 2.5, (ii) we can show the following assertions:

Theorem 2.3. For any c with $0 < c \leqq b$ and $q \geqq 1$, it holds that
(i) $O(S_c)(u_1, \ldots, u_k)$ has a structure of a finitely generated torsion free $O(S_c)$-module and represents $H^q(X(S_c), \Omega_X^n(E))$ as an $O(S_c)$-module.

(ii) $u \in C^{n,q}(X(S_c), E)$ satisfies $u \in O(S_c)(u_1, \ldots, u_k)$ if and only if $\bar{\partial} u = 0, \overline{\partial_h}^* u = 0$ and $\partial \varphi \wedge * u = 0$ on $X(S_c)$

(iii) the Hodge's operator relative to ω_X induces an splitting homomorphism

$$\delta^q : H^q(X(S_c), \Omega_X^n(E)) \to \Gamma(X(S_c), \Omega_X^{n-q}(E)),$$

with $L^q \circ \delta^q = id$ for the homomorphism

$$L^q : \Gamma(X(S_c), \Omega_X^{n-q}(E)) \to H^q(X(S_c), \Omega_X^n(E))$$

induced by the q-times exterior product by ω_X, which commutes with the restriction homomorphism of cohomology group.

(iv) $H^q(X(S_c), \Omega_X^n(E)) = 0$ for $q > n - m$.

The other results of Theorem is more or less derived from this theorem. The above harmonic representation is natural in the following sense.

Proposition 2.4. If S is non-singular, then for any c with $0 < c \leqq b$ and $0 \leqq q \leqq n - m$, it holds that the image of the homomorphism

$$L^q : \Gamma(X(S_c), \Omega_X^{n-m-q} \otimes f^* \Omega_S^m) \to H^q(X(S_c), \Omega_X^n)$$

is just contained in $O(S_c)(u_1, \dots, u_k)$. In particular L^q is injective and bijective if f has maximal rank over S_c.

Therefore it is a key point to show Lemma 2.2. This can be done by using an a-priori estimate for the $\bar{\partial}$-operator on any complete Kähler manifolds carrying bounded plurisubharmonic functions, which is derived from the formulae by Calabi-Vesentini and Donnelly-Xavier.

Proposition 2.5. Let (E, h) be a holomorphic vector bundle on an n dimensional Kähler manifold (X, ω_X) and let $\Omega := \{r < 0\}$ be a bounded domain with smooth boundary $\partial\Omega$ on X.
Then the following formulae hold :

(i)
$$\|\sqrt{\eta}(\bar\partial + e(\bar\partial\varphi))u\|_h^2 + \|\sqrt{\eta}\ \overline{\partial}_h^* u\|_E^2$$

$$= \|\sqrt{\eta}(\partial^* - e(\partial\varphi)^*)u\|_h^2 + \|\sqrt{\eta}D_h' u\|_h^2 + (\eta\, i[e(\Theta_h + \partial\bar\partial\varphi), \Lambda]u, u)_h$$

for any $u \in C_0^{p,q}(X, E)$ and any real-valued smooth function φ with $\eta := \exp\varphi$ on X

(ii)
$$\frac{\partial}{\partial t}[e(\bar\partial r)^* u]_{h,t}^2 = [i\, e(\partial\bar\partial r)\Lambda u, u]_{h,t} + \|\partial^* u\|_{h,t}^2 + (i\, e(\Theta_h)\Lambda u, u)_{h,t}$$

$$- 2\mathrm{Re}[\overline{\partial_h}^* u, e(\bar\partial r)^* u]_{h,t} - \|\bar\partial u\|_{h,t}^2 - \|\overline{\partial_h}^* u\|_{h,t}^2$$

for any $u \in C^{n,q}(X, E)$ and $0 \leqq |t| \ll 1$

Using those estimates we can show the following assertion:

Theorem 2.6. Let (X, ω_X) be an n dimensional complete Kähler manifold and let (E, h) be a Nakano semi-positive vector bundle on X. Then if $u \in L^{n,q}(X, E, \omega_X, h) \cap \mathrm{Ker}\,\square_h$, u satisfies the following equations:

(i) $\bar\partial u = 0, \overline{\partial_h}^* u = 0, \bar\partial * u = 0$ and $< i\ e\ (\Theta_h)\Lambda u, u >_h = 0$ on X

(ii) $\partial\varphi \wedge * u = 0$ and $< i\,\partial\bar\partial\varphi\ \Lambda u, u > = 0$ on X

for any smooth plurisubharmonic function φ on X satisfying $\sup_{x \in X} \{|\varphi(x)| + |d\varphi(x)|\} < \infty$

We can apply this theorem to our situation in the following way. Since $(X(S_a), \omega_X, \Phi := \frac{1}{a-\varphi})$ is a weakly 1-complete Kähler manifold,for the given forms $\{v_j\}$ we can construct a complete Kähler metric ω_* and a hermitian metric h_* of E on $X(S_a)$ such that (i) the restriction ω_* and h_* onto $X(S_b)$ coincide with the original metric ω_X and h respectively (ii) $v_j \in L^{n,q}(X(S_a), E, \omega_*, h^*)$ for any j. To see this we have only to take a smooth convex increasing function $\lambda(s)$ satisfying $\lambda(s) = 0$ for $s \leqq b$ arbitrarily and set $\omega_* = \omega_X + i\partial\bar{\partial}\lambda(\Phi)$ and $h_* = h\exp(-\lambda(\Phi))$.

Since each v_j is $\bar{\partial}$-closed,we have $v_j \in [Range\,\bar{\partial}] \oplus H^{n,q}(E)$ $(H^{n,q}(E) := \operatorname{Ker} \square_{h*} \cap L^{n,q}(X(S_a), E, \omega_*, h^*))$. Hence we take the orthogonal projection of v_j onto $H^{n,q}(E)$ and denote it by u_j.By the construction of ω_* and h^* each u_j satisfies Lemma 2.2,(ii) and (iii) on $X(S_b)$ by Proposition 2. On the other hand we have $v_j - u_j \in [Range\,\bar{\partial}]$.However by the separability of the cohomology group (cf.(2.1)) we can find a smooth form w_j so that $v_j = u_j + \bar{\partial}w_j$ for each j. This completes the proof of Lemma 2.2.

References

[G-R],Grauert,H.& Riemennschneider,O., Verschwindungssätze für analytische Kohomologiegruppen auf komplexen Raumen, Invent. Math. 11 (1970),263-292

[Ko-1], [Ko-2], Kollár, J., Higher direct images of dualizing sheaves I, II., Ann. Math. 123 (1986), 11-42, 124 (1986), 171-202

[Mo], Moriwaki, A., Torsion freeness of higher direct images of canonical bundles, Math. Ann. 276 (1987), 385-398

[Nm], Nakamura, I., Complex parallelisable manifolds and their smoll deformations, J. Differential Geometry, 10 (1975), 85-112

[Ny], Nakayama, N., Hodge filtrations and the higher direct images of canonical sheaves, Invent. Math. 86 (1986), 217-221

[Oh], Ohsawa, T., Vanishing theorems on complete Kähler manifolds, Publ. RIMS. Kyoto Univ. 20 (1984), 21-38

[Sa], Saito, M., Modules de Hodge polarisables, Publ. RIMS. Kyoto Univ. 24 (1988), 849-995